Introduction to Engineering Optimization (ME6806)

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Constrained Optimization Algorithms: Kuhn-Tucker Conditions

Kuhn-Tucker Conditions

Consider the following optimization problem:

Minimize $f(\mathbf{X})$ subject to $g_j(X) \leq 0$ for $j = 1,2,\ldots,p$; where $\mathbf{X} = [x_1 \; x_2 \ldots x_n]$

Then the Kuhn-Tucker conditions for $\mathbf{X}^* = [x_1^* \; x_2^* \; \ldots x_n^*]$ to be a local minimum are

$$\begin{aligned}
\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g}{\partial x_i} &= 0 \qquad i = 1, 2, \ldots, n \\
\lambda_j g_j &= 0 \qquad j = 1, 2, \ldots, m \\
g_j &\leq 0 \qquad j = 1, 2, \ldots, m \\
\lambda_j &\geq 0 \qquad j = 1, 2, \ldots, m
\end{aligned} \tag{1}$$

Kuhn-Tucker Conditions - Example

Minimize $f = x_1^2 + 2x_2^2 + 3x_3^2$ subject to the constraints

$$g_1 = x_1 - x_2 - 2x_3 \le 12$$

$$g_2 = x_1 + 2x_2 - 3x_3 \le 8$$

using Kuhn-Tucker conditions.

a) $$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0$$

i.e.

$$2x_1 + \lambda_1 + \lambda_2 = 0 \qquad (2)$$

$$4x_2 - \lambda_1 + 2\lambda_2 = 0 \qquad (3)$$

$$6x_3 - 2\lambda_1 - 3\lambda_2 = 0 \qquad (4)$$

b) $$\lambda_j g_j = 0$$

i.e.

$$\lambda_1(x_1 - x_2 - 2x_3 - 12) = 0 \qquad (5)$$

$$\lambda_2(x_1 + 2x_2 - 3x_3 - 8) = 0 \qquad (6)$$

Kuhn-Tucker Conditions - Example

c) $g_j \leq 0$

i.e.,

$$x_1 - x_2 - 2x_3 - 12 \leq 0 \qquad (7)$$

$$x_1 + 2x_2 - 3x_3 - 8 \leq 0 \qquad (8)$$

d) $\lambda_j \geq 0$

i.e.,

$$\lambda_1 \geq 0 \qquad (9)$$

$$\lambda_2 \geq 0 \qquad (10)$$

From (5) either $\lambda_1 = 0$ or, $x_1 - x_2 - 2x_3 - 12 = 0$

Case 1: $\lambda_1 = 0$

From *(2), (3) and (4)* we have $x_1 = x_2 = -\lambda_2 / 2$ and $x_3 = \lambda_2 / 2$.

Using these in *(6)* we get $\lambda_2^2 + 8\lambda_2 = 0, \therefore \lambda_2 = 0 \; or \; -8$

From *(10)*, $\lambda_2 \geq 0$, therefore, $\lambda_2 = 0$, $\mathbf{X}^* = [\, 0, 0, 0 \,]$,

this solution set satisfies all of *(6)* to *(9)*

Kuhn-Tucker Conditions - Example

Case 2: $x_1 - x_2 - 2x_3 - 12 = 0$

Using *(2), (3) and (4)*, we have $\frac{-\lambda_1 - \lambda_2}{2} - \frac{\lambda_1 - 2\lambda_2}{4} - \frac{2\lambda_1 + 3\lambda_2}{3} - 12 = 0$ or,

$17\lambda_1 + 12\lambda_2 = -144$. But conditions *(9)* and *(10)* give us $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ simultaneously,

which cannot be possible with $17\lambda_1 + 12\lambda_2 = -144$.

Hence the solution set for this optimization problem is $\mathbf{X}^* = [\ 0\ 0\ 0\ ]$

Practice Problem

Minimize $f = x_1^2 + x_2^2 + 60x_1$ subject to the constraints

$$g_1 = x_1 - 80 \geq 0$$

$$g_2 = x_1 + x_2 - 120 \geq 0$$

using Kuhn-Tucker conditions.

The solution set for this optimization problem is $\mathbf{X}^* = [\ 80\ 40\ ]$.