Introduction to Engineering Optimization (ME6806)

Dr. Yogesh Kumar
Assistant Professor
Mechanical Engineering Department
National Institute of Technology Patna
Bihar - 800 005, India
yogesh.me@nitp.ac.in

Module 4

Multi-variable Optimization Algorithms

Outlines

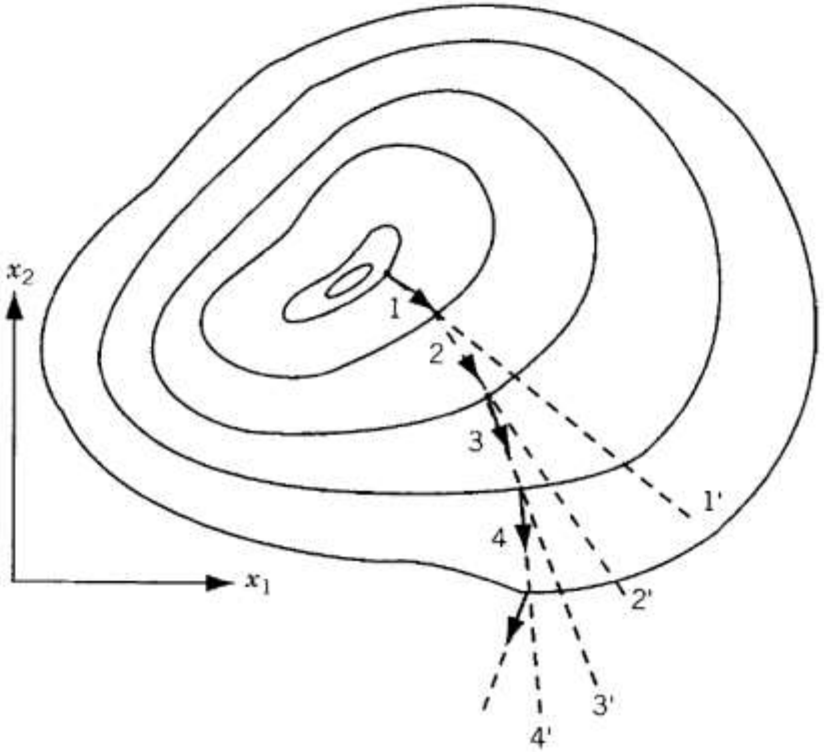
- **Optimality Criteria**
- **Direct Search Methods**
    - Nelder and Mead (Simplex Search)
    - Hook and Jeeves (Pattern Search)
    - Powell’s Method (The Conjugated Direction Search)
- **Gradient Based Methods**
    - **Steepest Descent (Cauchy’s) Method**
    - Newton’s Method
    - Modified Newton’s Method
    - Marquardt’s Method
    - Conjugate Gradient Method
    - Quasi-Newton Method
    - Trust Regions
    - Gradient-Based Algorithm
    - Numerical Gradient Approximations

Gradient Based Methods

Gradient of Function

The gradient of a function is an *n*-component vector given by

$$\underset{n\times 1}{\nabla f} = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_n \end{Bmatrix}$$



Steepest Descent (Cauchy's) Method

1. Start with an arbitrary initial point $\mathbf{X}_1$. Set the iteration number as $i = 1$.
2. Find the search direction $\mathbf{S}_i$ as

$$\mathbf{S}_i = -\nabla f_i = -\nabla f(\mathbf{X}_i)$$

3. Determine the optimal step length λ_i^* in the direction $\mathbf{S}_i$ and set

$$\mathbf{X}_{i+1} = \mathbf{X}_i + \lambda_i^* \mathbf{S}_i = \mathbf{X}_i - \lambda_i^* \nabla f_i$$

4. Test the new point, $\mathbf{X}_{i+1}$, for optimality. If $\mathbf{X}_{i+1}$ is optimum, stop the process. Otherwise, go to step 5.
5. Set the new iteration number $i = i + 1$ and go to step 2.

Steepest Descent (Cauchy's) Method

Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ from the starting point $\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Solution

Iteration 1

The gradient of f is given by

$$\nabla f = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix} = \begin{Bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{Bmatrix}$$

$$\nabla f_1 = \nabla f(\mathbf{X}_1) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Therefore,

$$\mathbf{S}_1 = -\nabla f_1 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Steepest Descent (Cauchy's) Method

Therefore,

$$\mathbf{S}_1 = -\nabla f_1 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

To find $\mathbf{X}_2$, we need to find the optimal step length λ_1^*. For this, we minimize $f(\mathbf{X}_1 + \lambda_1 \mathbf{S}_1) = f(-\lambda_1, \lambda_1) = \lambda_1^2 - 2\lambda_1$ with respect to λ_1. Since $df/d\lambda_1 = 0$ at $\lambda_1^* = 1$, we obtain

$$\mathbf{X}_2 = \mathbf{X}_1 + \lambda_1^* \mathbf{S}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 1 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

As $\nabla f_2 = \nabla f(\mathbf{X}_2) = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $\mathbf{X}_2$ is not optimum.

Steepest Descent (Cauchy's) Method

Iteration 2

$$\mathbf{S}_2 = -\nabla f_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

To minimize

$$f(\mathbf{X}_2 + \lambda_2 \mathbf{S}_2) = f(-1 + \lambda_2, 1 + \lambda_2)$$
$$= 5\lambda_2^2 - 2\lambda_2 - 1$$

we set $df/d\lambda_2 = 0$. This gives $\lambda_2^* = \frac{1}{5}$, and hence

$$\mathbf{X}_3 = \mathbf{X}_2 + \lambda_2^* \mathbf{S}_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} + \frac{1}{5} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.8 \\ 1.2 \end{Bmatrix}$$

Since the components of the gradient at $\mathbf{X}_3$, $\nabla f_3 = \begin{Bmatrix} 0.2 \\ -0.2 \end{Bmatrix}$, are not zero, we proceed to the next iteration.

Steepest Descent (Cauchy's) Method

Iteration 3

$$\mathbf{S}_3 = -\nabla f_3 = \begin{Bmatrix} -0.2 \\ 0.2 \end{Bmatrix}$$

$$f(\mathbf{X}_3 + \lambda_3 \mathbf{S}_3) = f(-0.8 - 0.2\lambda_3, 1.2 + 0.2\lambda_3)$$

$$= 0.04\lambda_3^2 - 0.08\lambda_3 - 1.20, \quad \frac{df}{d\lambda_3} = 0 \text{ at } \lambda_3^* = 1.0$$

Therefore,

$$\mathbf{X}_4 = \mathbf{X}_3 + \lambda_3^* \mathbf{S}_3 = \begin{Bmatrix} -0.8 \\ 1.2 \end{Bmatrix} + 1.0 \begin{Bmatrix} -0.2 \\ 0.2 \end{Bmatrix} = \begin{Bmatrix} -1.0 \\ 1.4 \end{Bmatrix}$$

The gradient at $\mathbf{X}_4$ is given by

$$\nabla f_4 = \begin{Bmatrix} -0.20 \\ -0.20 \end{Bmatrix}$$

Since $\nabla f_4 \neq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $\mathbf{X}_4$ is not optimum and hence we have to proceed to the next iteration. This process has to be continued until the optimum point, $\mathbf{X}^* = \begin{Bmatrix} -1.0 \\ 1.5 \end{Bmatrix}$, is found.

Steepest Descent (Cauchy's) Method

Convergence Criteria: The following criteria can be used to terminate the iterative process.

1. When the change in function value in two consecutive iterations is small:

$$\left| \frac{f(\mathbf{X}_{i+1}) - f(\mathbf{X}_i)}{f(\mathbf{X}_i)} \right| \le \varepsilon_1$$

2. When the partial derivatives (components of the gradient) of f are small:

$$\left| \frac{\partial f}{\partial x_i} \right| \le \varepsilon_2, \quad i = 1, 2, \ldots, n$$

3. When the change in the design vector in two consecutive iterations is small:

$$|\mathbf{X}_{i+1} - \mathbf{X}_i| \le \varepsilon_3$$