Introduction to Engineering Optimization (ME6806)

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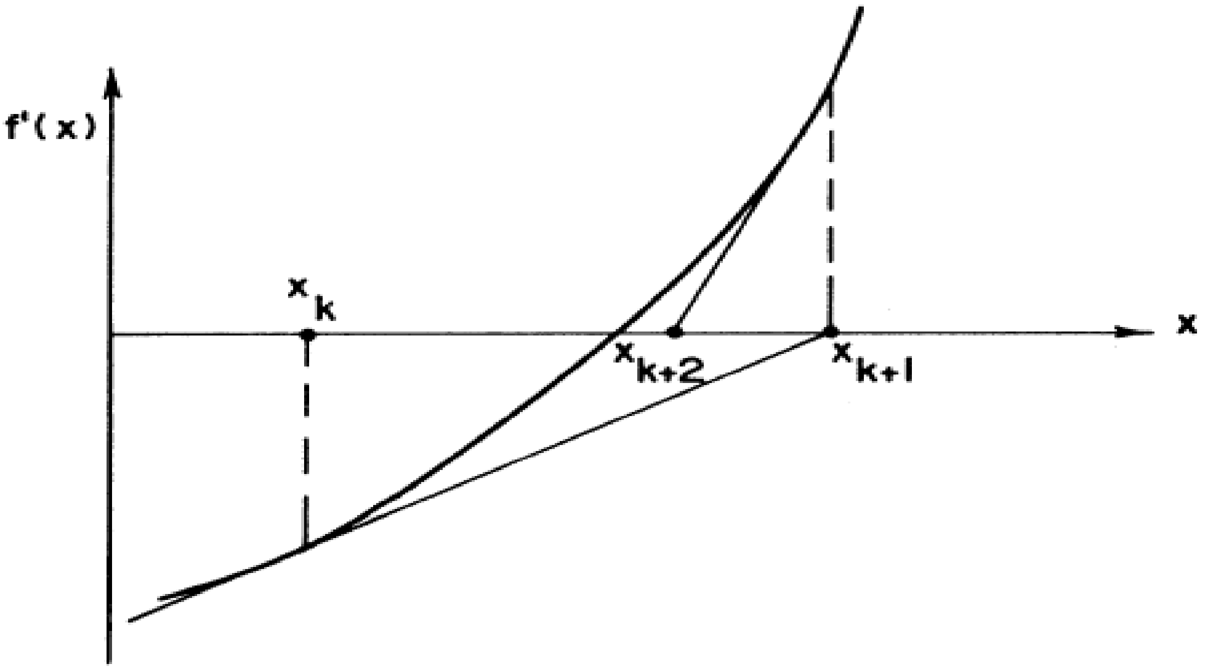
Methods Requiring Derivatives

- Newton–Raphson Method
- Bisection Method
- Secant Method
- Cubic Search Method

Newton-Raphson Method

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$



Newton-Raphson Method

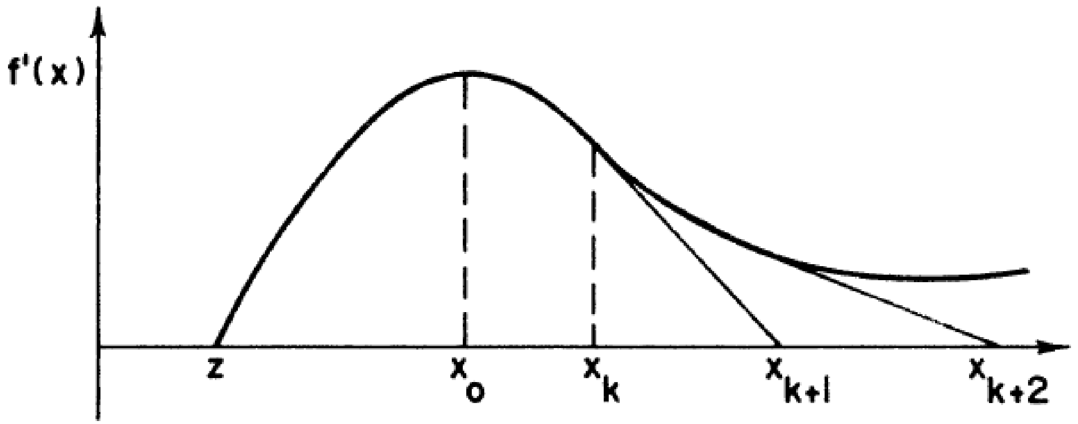
Consider the problem

$$\text{Minimize } f(x) = 2x^2 + \frac{16}{x}$$

Suppose we use the Newton–Raphson method to determine a stationary point of $f(x)$ starting at the point $x_1 = 1$:

$$f'(x) = 4x - \frac{16}{x^2} \qquad f''(x) = 4 + \frac{32}{x^3}$$

Newton-Raphson Method



Step 1. $x_1 = 1 \qquad f'(x_1) = -12 \qquad f''(x_1) = 36$

$$x_2 = 1 - \frac{-12}{36} = 1.33$$

Step 2. $x_2 = 1.33 \qquad f'(x_2) = -3.73 \qquad f''(x_2) = 17.6$

$$x_3 = 1.33 - \frac{-3.73}{17.6} = 1.54$$

We continue until $|f'(x_k)| < \varepsilon$, where ε is a prespecified tolerance.

Bisection Method - Introduction

- Bisection Method = a numerical method in Mathematics to find a root of a given *function*

- Root of a function $f(x)$ = a value a such that:

$$f(a) = 0$$

Function: $f(x) = x^2 - 4$

Roots: $x = -2, \; x = 2$

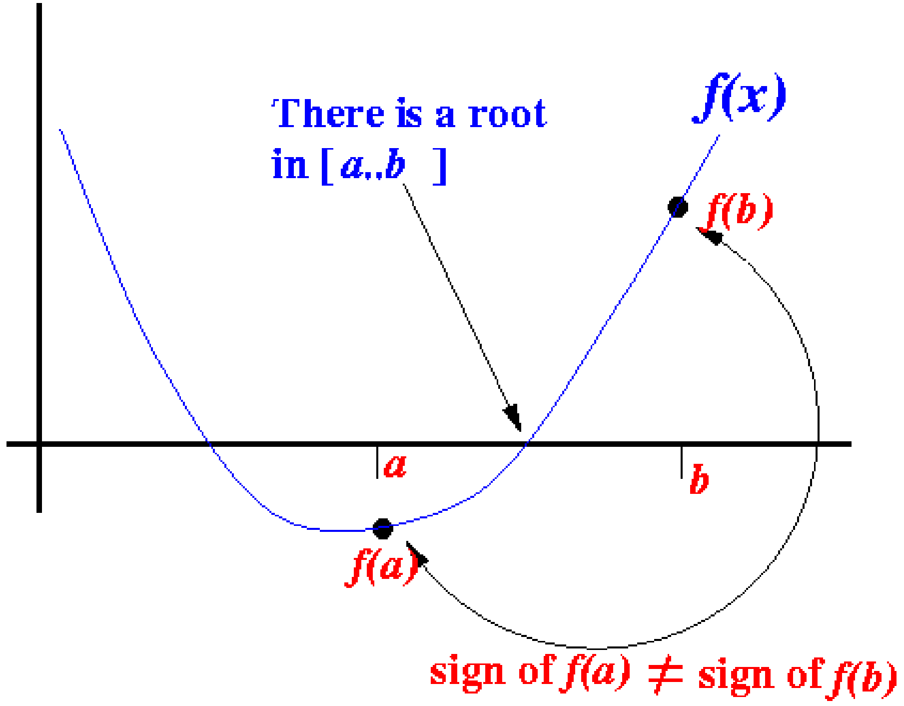
Because:

$$f(-2) = (-2)^2 - 4 = 4 - 4 = 0$$
$$f(2) = (2)^2 - 4 = 4 - 4 = 0$$

Bisection Method - Introduction

- If a function $f(x)$ is continuous on the interval $[a..b]$ and sign of $f(a) \neq$ sign of $f(b)$, then:

  - There is a value $c \in [a..b]$ such that: $f(c) = 0$ i.e., there is a root c in the interval $[a..b]$



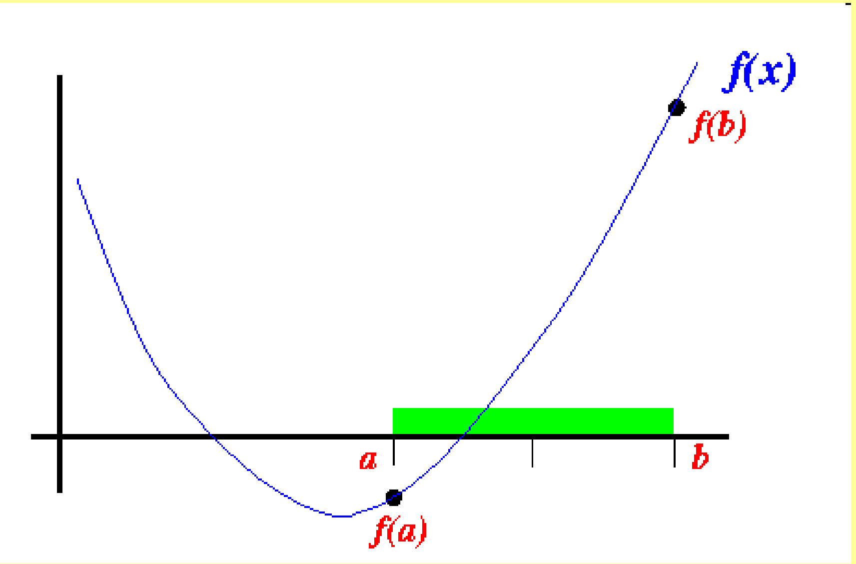
Bisection Method - Introduction

- The Bisection Method is a *successive* approximation method that narrows down an interval that contains a root of the function $f(x)$.
- The Bisection Method is *given* an initial interval $[a..b]$ that contains a root (We can use the property sign of $f(a) \neq$ sign of $f(b)$ to find such an initial interval).
- The Bisection Method will *cut the interval* into 2 halves and check which half interval contains a root of the function.
- The Bisection Method will keep *cut the interval* in halves until the resulting interval is extremely small.

  The root is then *approximately equal* to *any value* in the final (very small) interval.

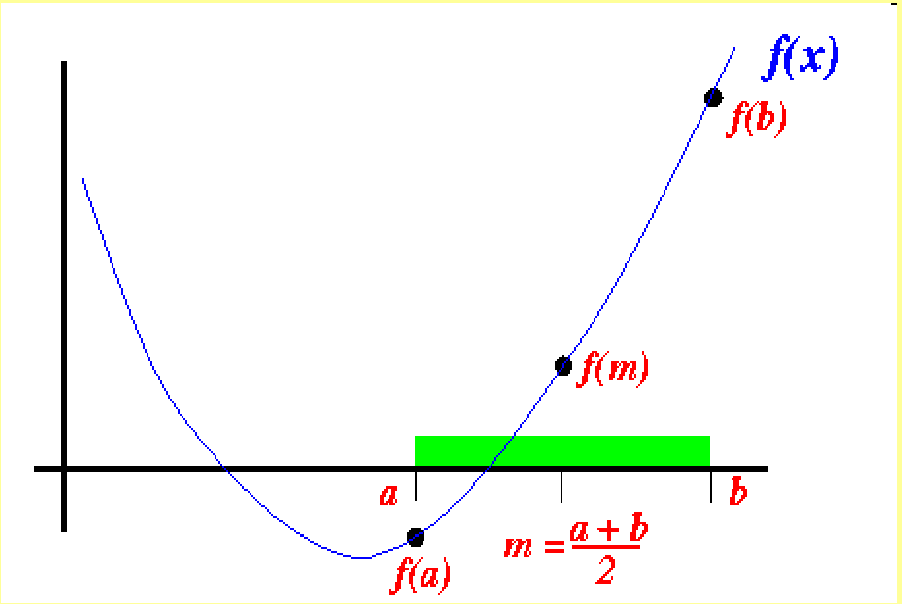
Bisection Method - Introduction

- Suppose the interval [*a*..*b*] is as follows:



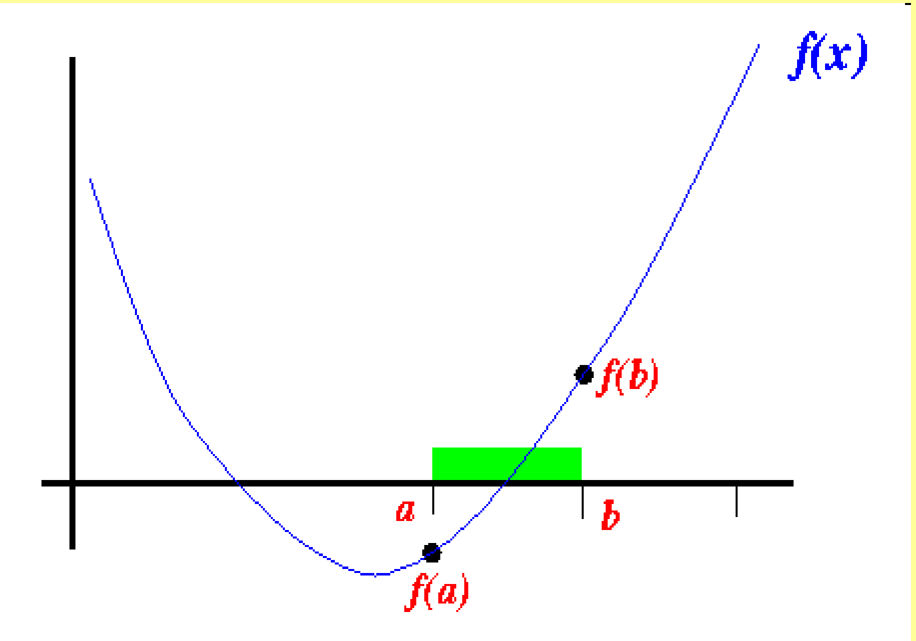
- We cut the interval [*a*..*b*] in the middle:

$$m = (a+b)/2$$



Bisection Method - Introduction

- Because sign of $f(m) \neq$ sign of $f(a)$, we *proceed* with the search in the *new* interval [*a*..*b*]:



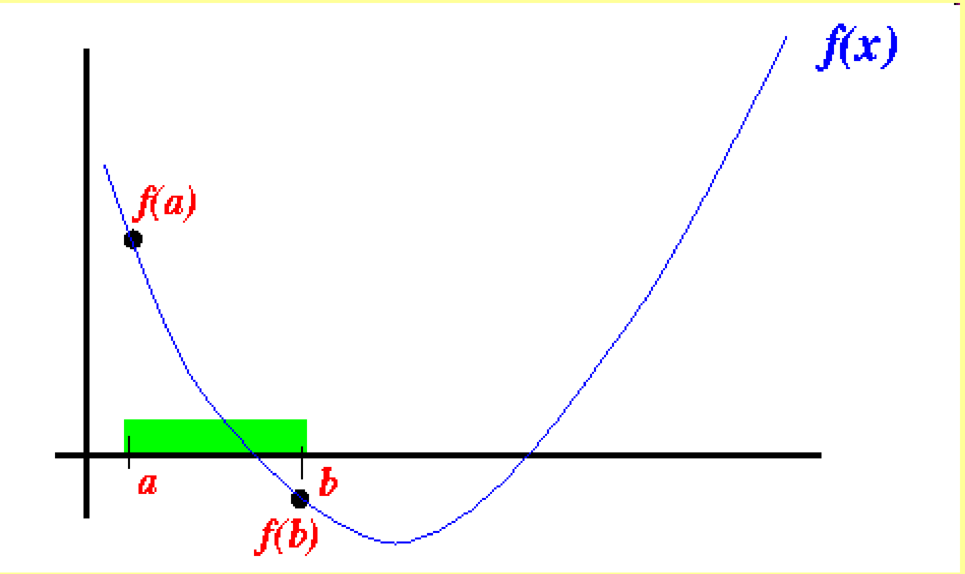
We can use this statement to change to the new interval:

b = m;

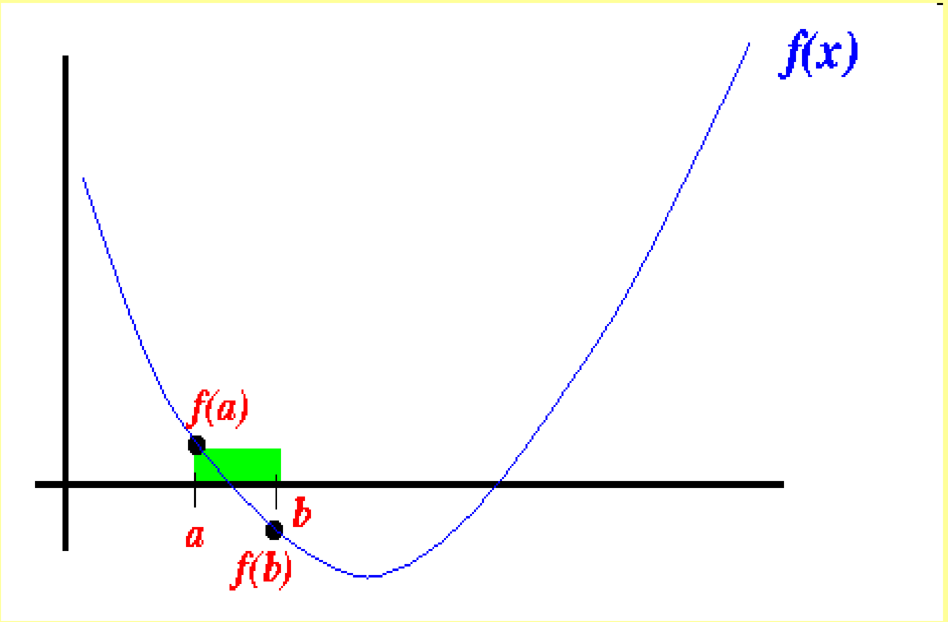
- In the this example, we have changed the end point *b* to obtain a smaller interval that still contains a root.
- In other cases, we may need to changed the end point *b* to obtain a smaller interval that still contains a root.

Bisection Method - Introduction

- Initial interval [*a*..*b*]:



- After cutting the interval in half, the root is contained in the right-half, so we have to change the end point *a*:



Bisection Method - Introduction

Given: interval [a..b] such that: sign of f(a) ≠ sign of f(b)

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repeat (until the interval [a..b] is "very small")
{
        a+b
   m = -----;      // m = midpoint of interval [a..b]
         2

   if ( sign of f(m) ≠ sign of f(b) )
   {
     use interval [m..b]  in the next iteration
     (i.e.: replace a with m)
   }
   else
   {
     use interval [a..m]  in the next iteration (i.e.:
     replace b with m)
   }
}
```

Approximate root = (a+b)/2;
(any point between [a..b] will do because the interval [a..b] is very small)

Bisection Method

Determine two points L and R such that $f'(L) < 0$ and $f'(R) > 0$. The stationary point is between the points L and R. We determine the derivative of the function at the midpoint,

$$z = \frac{L + R}{2}$$

If $f'(z) > 0$, then the interval (z, R) can be eliminated from the search. On the other hand, if $f'(z) < 0$, then the interval (L, z) can be eliminated. We shall now state the formal steps of the algorithm.

Given a bounded interval $a \leqslant x \leqslant b$ and a termination criterion ε:

Step 1. Set $R = b$, $L = a$; assume $f'(a) < 0$ and $f'(b) > 0$.

Step 2. Calculate $z = (R + L)/2$, and evaluate $f'(z)$.

Step 3. If $|f'(z)| \leqslant \varepsilon$, terminate. Otherwise, if $f'(z) < 0$, set $L = z$ and go to step 2. If $f'(z) > 0$, set $R = z$ and go to step 2.