Introduction to Engineering Optimization (ME6806)

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Module 3

Single-variable Optimization Algorithms

Algorithm to Find Global Optima

$$\text{Maximize} \quad f(x)$$
$$\text{Subject to} \quad a \leqslant x \leqslant b$$

Step 1. Set $df/dx = 0$ and compute all stationary points.

Step 2. Select all stationary points that belong to the interval $[a, b]$. Call them $x_1, x_2, \ldots, x_N$. These points, along with a and b, are the only points that can qualify for a local optimum.

Step 3. Find the largest value of $f(x)$ out of $f(a), f(b), f(x_1), \ldots, f(x_N)$. This value becomes the global maximum point.

An Example

Maximize $f(x) = -x^3 + 3x^2 + 9x + 10$ in the interval $-2 \leqslant x \leqslant 4$

$$\frac{df}{dx} = -3x^2 + 6x + 9 = 0$$

Stationary points x = -1, 3

To find the global maximum, evaluate $f(x)$ at $x = 3, -1, -2,$ and 4:

$$f(3) = 37 \qquad f(-1) = 5$$

$$f(-2) = 12 \qquad f(4) = 30$$

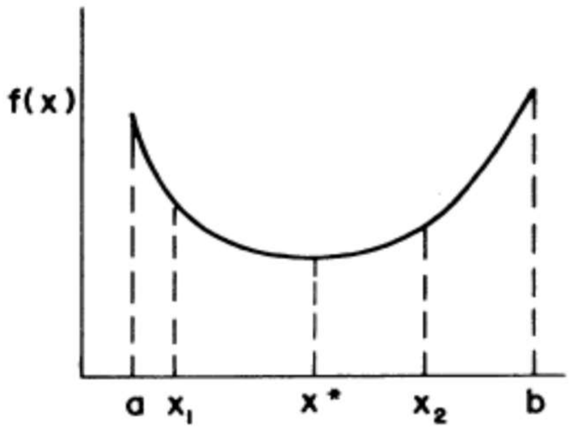
Hence $x = 3$ maximizes f over the interval $(-2, 4)$

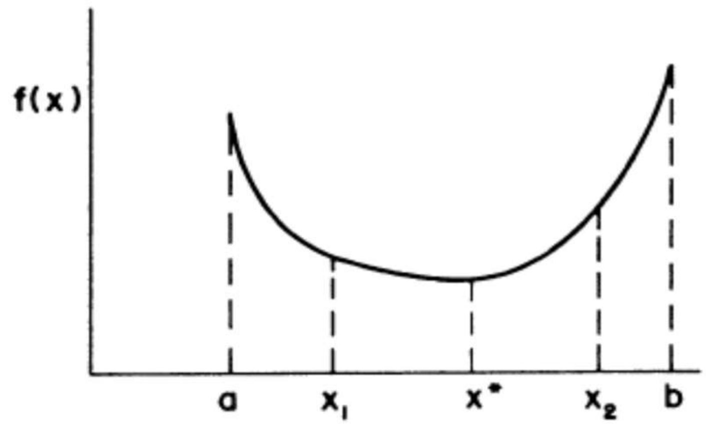
Region Elimination Methods

Suppose f is strictly unimodal[†] on the interval $a \leqslant x \leqslant b$ with a minimum at x^*. Let x_1 and x_2 be two points in the interval such that $a < x_1 < x_2 < b$. Comparing the functional values at x_1 and x_2, we can conclude:

(i) If $f(x_1) > f(x_2)$, then the minimum of $f(x)$ does not lie in the interval (a, x_1). In other words, $x^* \in (x_1, b)$

(ii) If $f(x_1) < f(x_2)$, then the minimum does not lie in the interval (x_2, b) or $x^* \in (a, x_2)$





Region Elimination Methods

- Bounding Phase
  - An initial coarse search that will bound or bracket the optimum
- Interval Refinement Phase
  - A finite sequence of interval reductions or refinements to reduce the initial search interval to desired accuracy

Bounding Phase

- Swann's method

$$x_{k+1} = x_k + 2^k\Delta \qquad \text{for } k = 0, 1, 2, \ldots$$

  - If

  $f(x_0 - |\Delta|) \geqslant f(x_0) \geqslant f(x_0 + |\Delta|)$ → Δ is positive

  - Else if the inequalities are reversed → Δ is negative

  - If $f(x_0 - |\Delta|) \geqslant f(x_0) \leqslant f(x_0 + |\Delta|)$ → the minimum lies between $x_0 - |\Delta|$ and $x_0 + |\Delta|$

Bounding Phase - Example

Consider the problem of minimizing $f(x) = (100 - x)^2$ given the starting point $x_0 = 30$ and a step size $|\Delta| = 5$.

The sign of Δ is determined by comparing

$$f(x_0) = f(30) = 4900$$

$$f(x_0 + |\Delta|) = f(35) = 4225$$

$$f(x_0 - |\Delta|) = f(25) = 5625$$

Since

$$f(x_0 - |\Delta|) \geqslant f(x_0) \geqslant f(x_0 + |\Delta|)$$

Δ must be positive, and the minimum point x^* must be greater than 30. Thus, $x_1 = x^0 + \Delta = 35$.

Bounding Phase - Example

Next,

$$x_2 = x_1 + 2\Delta = 45$$

$$f(45) = 3025 < f(x_1)$$

therefore, $x^* > 35$;

$$x_3 = x_2 + 2^2\Delta = 65$$

$$f(65) = 1225 < f(x_2)$$

therefore, $x^* > 45$;

$$x_4 = x_3 + 2^3\Delta = 105$$

$$f(105) = 25 < f(x_3)$$

Bounding Phase - Example

therefore, $x^* > 65$;

$$x_5 = x_4 + 2^4\Delta = 185$$

$$f(185) = 7225 > f(x_4)$$

therefore, $x^* < 185$. Consequently, in six evaluations x^* has been bracketed within the interval

$$65 \leqslant x^* \leqslant 185$$

Note that the effectiveness of the bounding search depends directly on the step size Δ. If Δ is large, a poor bracket, that is, a large initial interval, is obtained. On the other hand, if Δ is small, many evaluations may be necessary before a bound can be established.