Introduction to Engineering Optimization (ME6806)

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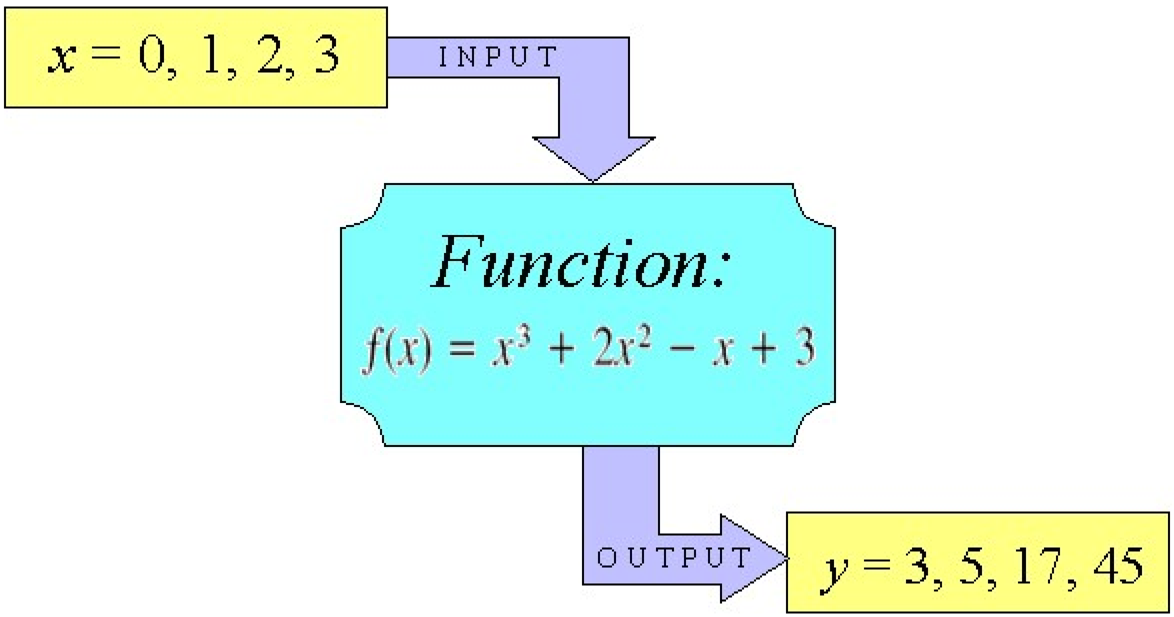
Single-variable Optimization Algorithms

What is a Function?



- Is a rule that assigns to every choice of x a unique value $y = f(x)$.
- **Domain** of a function is the set of all possible input values (usually x), which allows the function formula to work.
- **Range** is the set of all possible output values (usually y), which result from using the function formula.

What is a Function?

- Unconstrained and constrained function
  - Unconstrained: when domain is the entire set of real numbers *R*
  - Constrained: domain is a proper subset of *R*
- Continuous, discontinuous and discrete

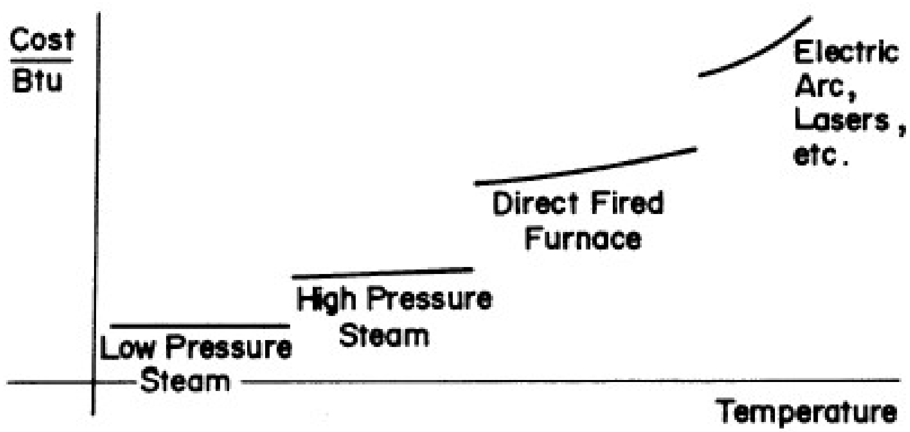


Figure 2.1. Discontinuous function.

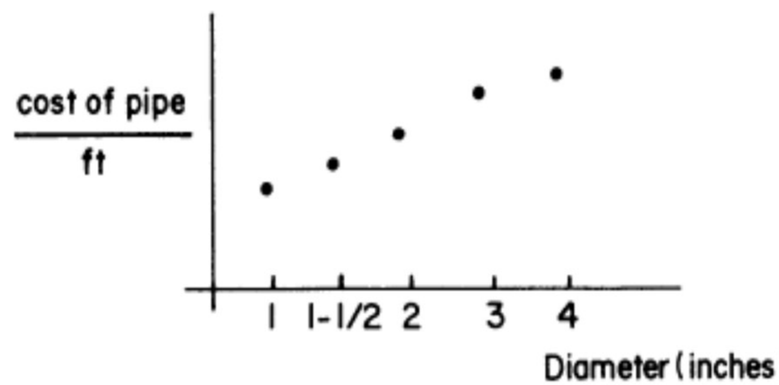


Figure 2.2. Discrete function.

What is a Function?

- Monotonic and unimodal functions
  - Monotonic:

    for any two points x_1 and x_2, with $x_1 \leqslant x_2$:

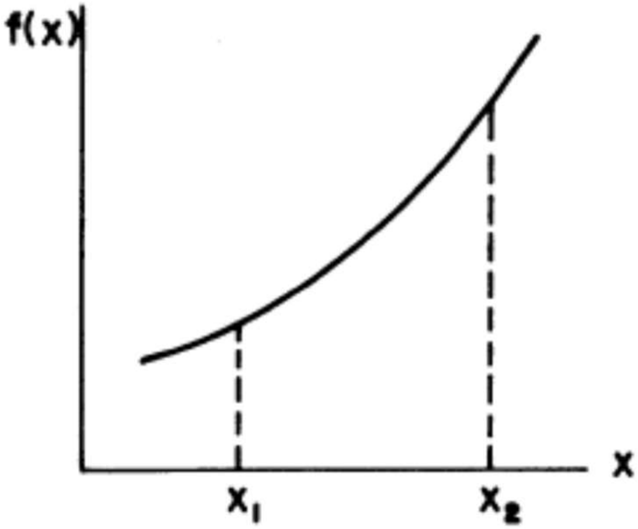
    $$f(x_1) \leqslant f(x_2) \quad \text{(monotonically increasing)}$$

    $$f(x_1) \geqslant f(x_2) \quad \text{(monotonically decreasing)}$$

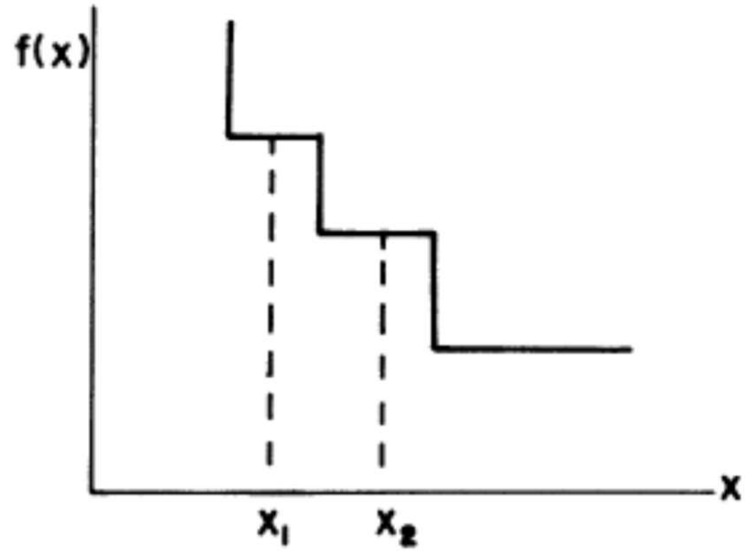
  - Unimodal:

    $f(x)$ is *unimodal* on the interval $a \leqslant x \leqslant b$ if and only if it is monotonic on either side of the single optimal point x^* in the interval.

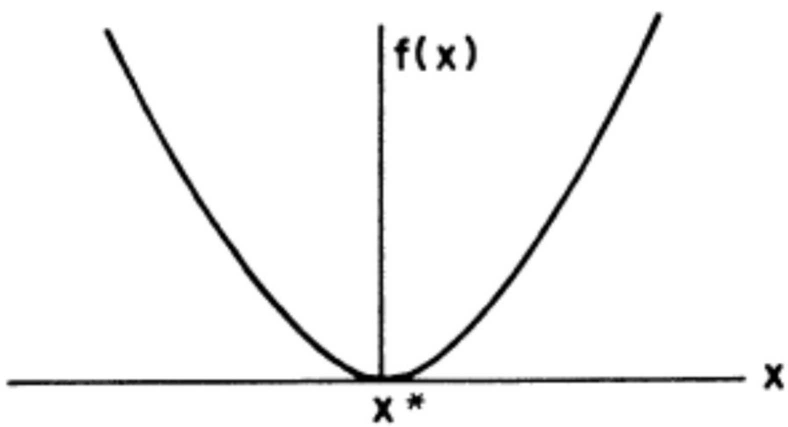
Unimodality is an extremely important functional property used in optimization.



A monotonic increasing function



A monotonic decreasing function



An unimodal function

Optimality Criteria

- In considering optimization problems, two questions generally must be addressed:

1. **Static Question.** How can one determine whether a given point x* is the optimal solution?
2. **Dynamic Question.** If x* is not the optimal point, then how does one go about finding a solution that is optimal?

Optimality Criteria

- Local and global optimum

A function $f(x)$ defined on a set S attains its *global minimum* at a point $x^{**} \in S$ if and only if

$$f(x^{**}) \leqslant f(x) \qquad \text{for all } x \in S$$

A function $f(x)$ defined on S has a *local minimum* (*relative minimum*) at a point $x^{*} \in S$ if and only if

$$f(x^{*}) \leqslant f(x) \qquad \text{for all } x \text{ within a distance } \varepsilon \text{ from } x^{*}$$

that is, there exists an $\varepsilon > 0$ such that, for all x satisfying $|x - x^{*}| < \varepsilon$, $f(x^{*}) \leqslant f(x)$.

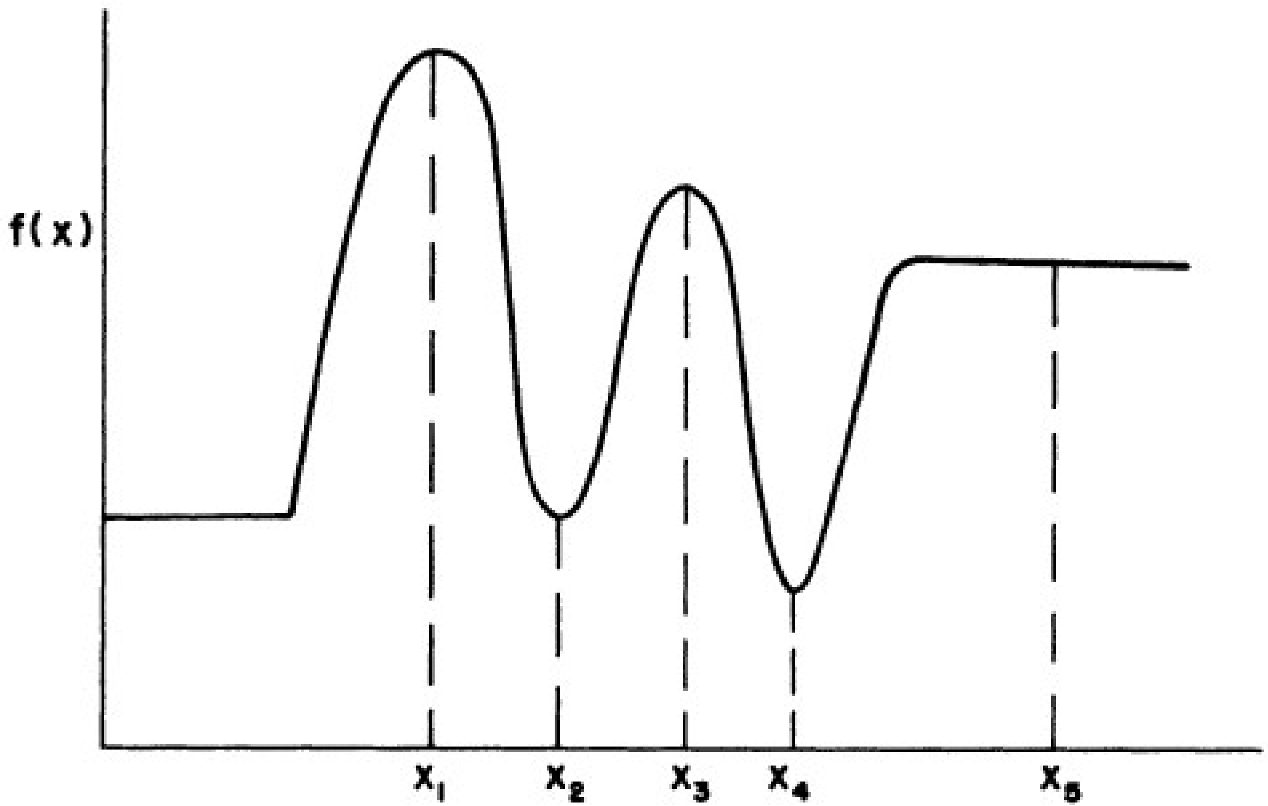


Figure 2.7. Local and global optima.

Identification of Single-Variable Optima

- For finding local minima (maxima)

$$\left.\frac{df}{dx}\right|_{x=x^*} = 0 \qquad \text{AND} \qquad \left.\frac{d^2f}{dx^2}\right|_{x=x^*} \geqslant 0 \ (\leqslant 0)$$

- Proof follows…

- These are necessary conditions, i.e., if they are not satisfied, x^* is not a local minimum (maximum). If they are satisfied, we still have no guarantee that x^* is a local minimum (maximum).

Stationary Point and Inflection Point

- A stationary point is a point x^* at which

$$\left.\frac{df}{dx}\right|_{x=x^*} = 0$$

- An *inflection point* or *saddle-point* is a stationary point that does not correspond to a local optimum (minimum or maximum).

- To distinguish whether a stationary point is a local minimum, a local maximum, or an inflection point, we need the *sufficient* conditions of optimality.

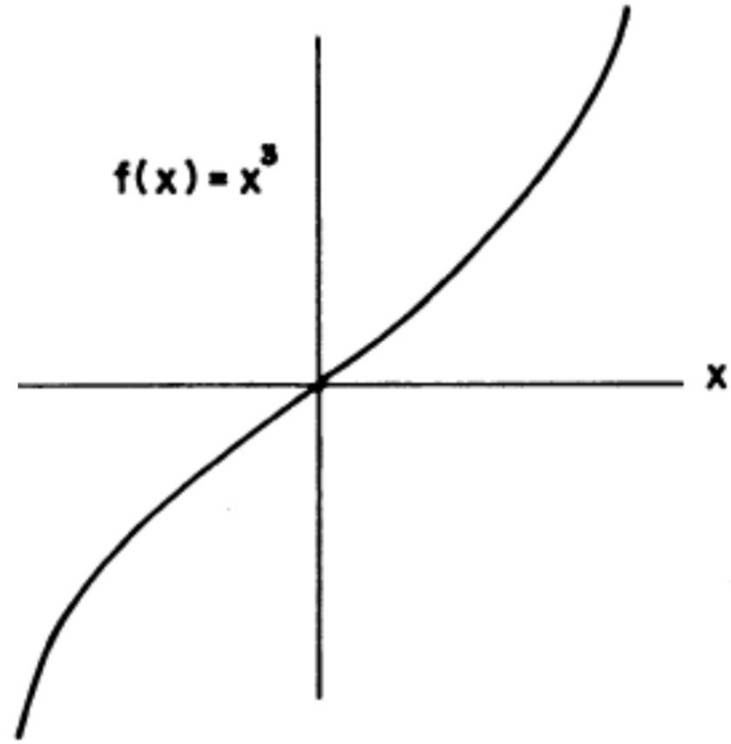
Theorem

- Suppose at a point x^* the first derivative is zero and the first nonzero higher order derivative is denoted by n.
  - If n is odd, then x^* is a point of inflection.
  - If n is even, then x^* is a local optimum.
- Moreover:
  - If that derivative is positive, then the point x^* is a local minimum.
  - If that derivative is negative, then the point x^* is a local maximum.

An Example

$$f(x) = x^3$$

$$\left.\frac{df}{dx}\right|_{x=0} = 0 \qquad \left.\frac{d^2f}{dx^2}\right|_{x=0} = 0 \qquad \left.\frac{d^3f}{dx^3}\right|_{x=0} = 6$$



- Thus the first non-vanishing derivative is 3 (odd), and $x = 0$ is an inflection point.