Assignment Problem using Hungarian Algorithm

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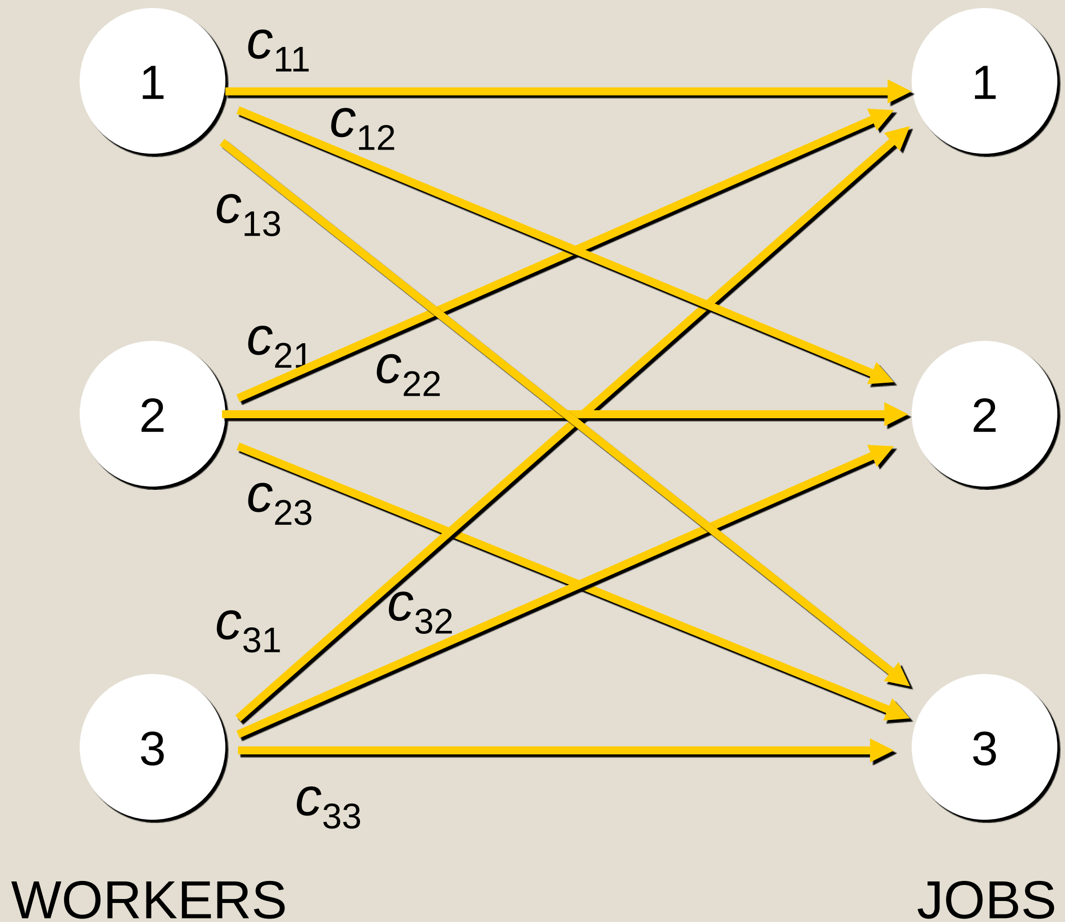
Assignment Problem

- An assignment problem seeks to minimize the total cost assignment of m workers to m jobs, given that the cost of worker i performing job j is c_{ij}.
- It assumes all workers are assigned and each job is performed.
- An assignment problem is a special case of a transportation problem in which all supplies and all demands are equal to 1; hence assignment problems may be solved as linear programs.
- The graph representation of an assignment problem with three workers and three jobs is shown on the next slide.

Assignment Problem

- Graph (or Network) Representation



Hungarian Method

- The Hungarian method solves minimization assignment problems with *m* workers and *m* jobs.
- Special considerations can include:
  - number of workers does not equal the number of jobs — add dummy workers/jobs with 0 assignment costs as needed
  - worker *i* cannot do job *j* — assign $c_{ij} = +M$
  - maximization objective — create an opportunity loss matrix subtracting all profits for each job from the maximum profit for that job before beginning the Hungarian method

Hungarian Method

- Step 1: For each row, subtract the minimum number in that row from all numbers in that row.
- Step 2: For each column, subtract the minimum number in that column from all numbers in that column.
- Step 3: Draw the minimum number of lines to cover all zeroes. If this number = m, STOP — an assignment can be made.
- Step 4: Determine the minimum uncovered number (call it d).
  - Subtract d from uncovered numbers.
  - Add d to numbers covered by two lines.
  - Numbers covered by one line remain the same.
  - Then, GO TO STEP 3.

Hungarian Method

- Finding the Minimum Number of Lines and Determining the Optimal Solution
  - Step 1: Find a row or column with only one unlined zero and circle it. (If all rows/columns have two or more unlined zeroes choose an arbitrary zero.)
  - Step 2: If the circle is in a row with one zero, draw a line through its column. If the circle is in a column with one zero, draw a line through its row. One approach, when all rows and columns have two or more zeroes, is to draw a line through one with the most zeroes, breaking ties arbitrarily.
  - Step 3: Repeat step 2 until all circles are lined. If this minimum number of lines equals m, the circles provide the optimal assignment.

Example 1: AP

A contractor pays his subcontractors a fixed fee plus mileage for work performed. On a given day the contractor is faced with three electrical jobs associated with various projects. Given below are the distances between the subcontractors and the projects.

		Project		
		A	B	C
Subcontractors	Westside	50	36	16
	Federated	28	30	18
	Goliath	35	33	20

How should the contractors be assigned to minimize total costs?

Example 1: AP

- Step 1: Subtract minimum number in each row from all numbers in that row.

	A	B	C
Westside	50	36	16
Federated	28	30	18
Goliath	35	33	20

This yields:

	A	B	C
Westside	34	20	0
Federated	10	12	0
Goliath	15	13	0

Example 1: AP

- Step 2: Subtract the minimum number in each column from all numbers in the column. For A it is 10, for B it is 12, for C it is 20. This yields:

	A	B	C
Westside	24	8	0
Federated	0	0	0
Goliath	5	1	0

Example 1: AP

- Step 3: Draw the minimum number of lines to cover all zeroes (called minimum cover). Although one can "eyeball" this minimum, use the following algorithm. If a "remaining" row has only one zero, draw a line through the column. If a remaining column has only one zero in it, draw a line through the row. Since the number of lines that cover all zeros is 2 < 3 (# of rows), the current solution is not optimal.

	A	B	C
Westside	24	8	0
Federated	0	0	0
Goliath	5	1	0

- Step 4: The minimum uncovered number is 1 (circled).

Example 1: AP

- Step 5: Subtract 1 from uncovered numbers; add 1 to all numbers at line intersections; leave all other numbers intact. This gives:

	A	B	C
Westside	23	7	0
Federated	0	0	1
Goliath	4	0	0

Example 1: AP

Find the minimum cover:

	A	B	C
Westside	23	7	0
Federated	0	0	1
Goliath	4	0	0

- Step 4: The minimum number of lines to cover all 0's is three. Thus, the current solution is optimal (minimum cost) assignment.

Example 1: AP

The optimal assignment occurs at locations of zeros such that there is exactly one zero in each row and each column:

	A	B	C
Westside	23	7	0
Federated	0	0	1
Goliath	4	0	0

Example 1: AP

The optimal assignment is (go back to the original table for the distances):

Subcontractor	Project	Distance
Westside	C	16
Federated	A	28
Goliath	B	33

Total Distance = 77 miles

Assignment Problem (Unbalance Problems) using Hungarian Algorithm

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Example 2: AP

A contractor pays his subcontractors a fixed fee plus mileage for work performed. On a given day the contractor is faced with three electrical jobs associated with various projects. Given below are the distances between the subcontractors and the projects.

		Project		
		A	B	C
Subcontractors	Westside	50	36	16
	Federated	28	30	18
	Goliath	35	33	20
	Universal	25	25	14

How should the contractors be assigned to minimize total costs?

Note: There are four subcontractors and three projects. We create a dummy project Dum, which will be assigned to one subcontractor (i.e. that subcontractor will remain idle)

Example 2: AP

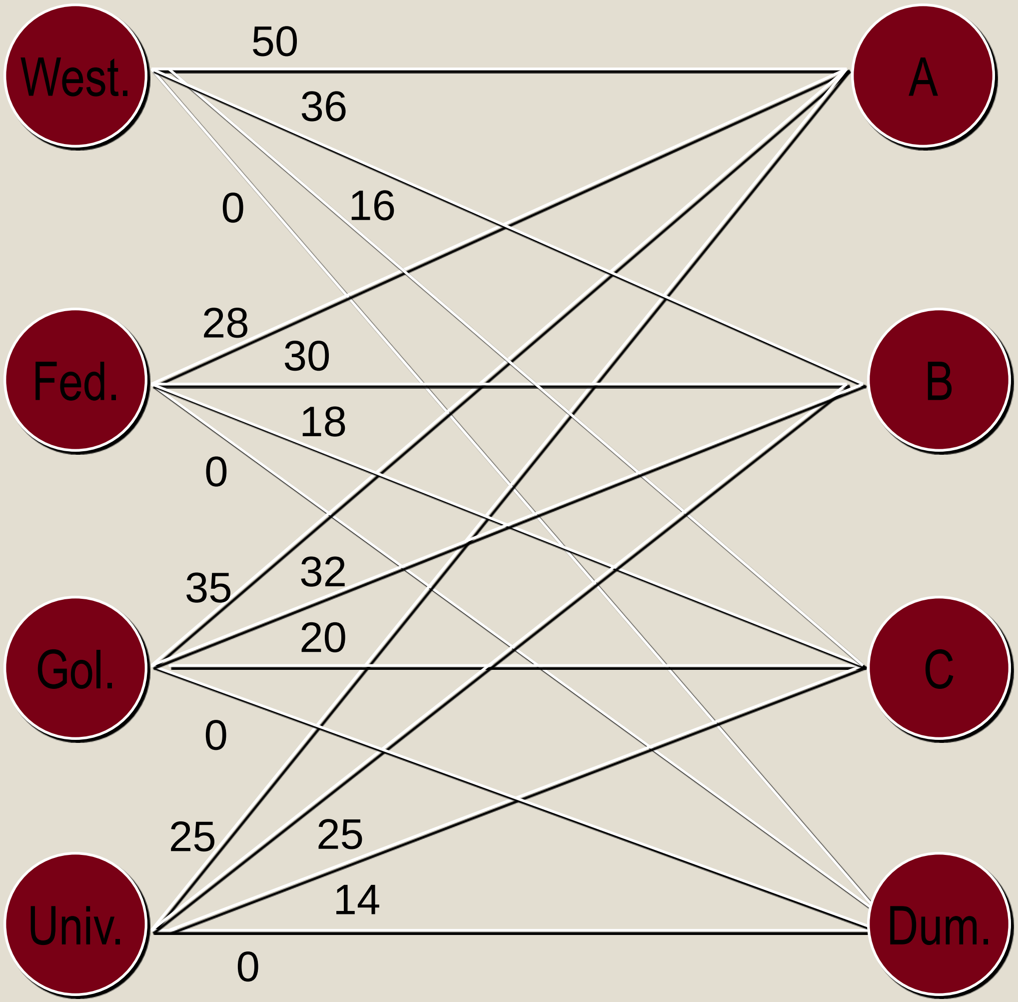
- Initial Tableau Setup

  Since the Hungarian algorithm requires that there be the same number of rows as columns, add a Dummy column so that the first tableau is (the smallest elements in each row are marked red):

	A	B	C	Dummy
Westside	50	36	16	0
Federated	28	30	18	0
Goliath	35	33	20	0
Universal	25	25	14	0

Example 2: AP

- Network Representation (note the dummy project)



Assignment Problem

- Linear Programming Formulation

$$\text{Min} \sum_{i}\sum_{j} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{j} x_{ij} = 1 \qquad \text{for each resource (row) } i$$

$$\sum_{i} x_{ij} = 1 \qquad \text{for each job (column) } j$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for all } i \text{ and } j.$$

  - Note: A modification to the right-hand side of the first constraint set can be made if a worker is permitted to work more than 1 job.

Example 2: AP via LP

- In our example the LP formulation is:

  Min $z = 50x_{11} + 36x_{12} + 16x_{13} + 0x_{14} + 28x_{21} + 30x_{22} + 18x_{23} + 0x_{24} + 35x_{31} + 32x_{32} + 20x_{33} + 0x_{34} + + 25x_{41} + 25x_{42} + 14x_{43} + 0x_{44}$

  s.t.

  $x_{11} + x_{12} + x_{13} + x_{14} = 1$ (row 1)

  $x_{21} + x_{22} + x_{23} + x_{24} = 1$ (row 2)

  $x_{31} + x_{32} + x_{33} + x_{34} = 1$ (row 3)

  $x_{41} + x_{42} + x_{43} + x_{44} = 1$ (row 4)

  $x_{11} + x_{21} + x_{31} + x_{41} = 1$ (column 1)

  $x_{12} + x_{22} + x_{32} + x_{42} = 1$ (column 2)

  $x_{13} + x_{23} + x_{33} + x_{43} = 1$ (column 3)

  $x_{14} + x_{24} + x_{34} + x_{44} = 1$ (column 4)

  $x_{ij} >= 0$ for i = 1, 2, 3, 4 and j = 1, 2, 3, 4 (nonnegativity)

Example 2: AP

- Step 1: Subtract minimum number in each row from all numbers in that row. Since each row has a zero, we simply generate the original matrix (the smallest elements in each column are marked red). This yields:

	A	B	C	Dummy
Westside	50	36	16	0
Federated	28	30	18	0
Goliath	35	33	20	0
Universal	25	25	14	0

Example 2: AP

- Step 2: Subtract the minimum number in each column from all numbers in the column. For A it is 25, for B it is 25, for C it is 14, for Dummy it is 0. This yields:

	A	B	C	Dummy
Westside	25	11	2	0
Federated	3	5	4	0
Goliath	10	8	6	0
Universal	0	0	0	0

Example 2: AP

- Step 3: Draw the minimum number of lines to cover all zeroes (called minimum cover). Although one can "eyeball" this minimum, use the following algorithm. If a "remaining" row has only one zero, draw a line through the column. If a remaining column has only one zero in it, draw a line through the row. Since the number of lines that cover all zeros is 2 < 4 (# of rows), the current solution is not optimal.

	A	B	C	Dummy
Westside	25	11	2	0
Federated	3	5	4	0
Goliath	10	8	6	0
Universal	0	0	0	0

- Step 4: The minimum uncovered number is 2 (circled).

Example 2: AP

- Step 5: Subtract 2 from uncovered numbers; add 2 to all numbers at line intersections; leave all other numbers intact. This gives:

	A	B	C	Dummy
Westside	23	9	0	0
Federated	1	3	2	0
Goliath	8	6	4	0
Universal	0	0	0	2

Example 2: AP

- Step 3: Draw the minimum number of lines to cover all zeroes. Since 3 (# of lines) < 4 (# of rows), the current solution is not optimal.

	A	B	C	Dummy
Westside	23	9	0	0
Federated	1	3	2	0
Goliath	8	6	4	0
Universal	0	0	0	2

- Step 4: The minimum uncovered number is 1 (circled).

Example 2: AP

- Step 5: Subtract 1 from uncovered numbers. Add 1 to numbers at intersections. Leave other numbers intact. This gives:

	A	B	C	Dummy
Westside	23	9	0	1
Federated	0	2	1	0
Goliath	7	5	3	0
Universal	0	0	0	3

Example 2: AP

Find the minimum cover:

	A	B	C	Dummy
Westside	23	9	0	1
Federated	0	2	1	0
Goliath	7	5	3	0
Universal	0	0	0	3

- Step 4: The minimum number of lines to cover all 0's is four. Thus, the current solution is optimal (minimum cost) assignment.

Example 2: AP

The optimal assignment occurs at locations of zeros such that there is exactly one zero in each row and each column:

	A	B	C	Dummy
Westside	23	9	0	1
Federated	0	2	1	0
Goliath	7	5	3	0
Universal	0	0	0	3

Example 2: AP

The optimal assignment is (go back to the original table for the distances):

Subcontractor	Project	Distance
Westside	C	16
Federated	A	28
Universal	B	25
Goliath	(unassigned)	

Total Distance = 69 miles